

Higgs Multiplets in Heterotic GUT Models

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Abstract

For supersymmetric GUT models from heterotic string theory, built from a stable holomorphic $SU(n)$ vector bundle V on a Calabi-Yau threefold X , the net amount of chiral matter can be computed by a Chern class computation. Corresponding computations for the number N_H of Higgses lead for the phenomenologically relevant cases of GUT group $SU(5)$ or $SO(10)$ to consideration of the bundle $\Lambda^2 V$. In a class of bundles where everything can be computed explicitly (spectral bundles on elliptic X) we find that the computation for N_H gives a result which is in conflict with expectations. We argue that this discrepancy has its origin in the subtle geometry of the spectral data for $\Lambda^2 V$ and that taking this subtlety into account properly should resolve the problem.

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1 Introduction

For supersymmetric GUT models from heterotic string theory, built from a stable holomorphic vector bundle V on a Calabi-Yau threefold X , the net amount of chiral matter can be computed by a Chern class computation. Corresponding computations for the number N_H of Higgses lead for the phenomenologically relevant cases of GUT group $SU(5)$ or $SO(10)$ to consideration of the bundle $\Lambda^2 V$. In a class of bundles where everything can be computed explicitly (spectral bundles on elliptic X) we find that the computation for N_H gives a result which is in conflict with expectations. We argue that this discrepancy has its origin in the subtle geometry of the spectral data for $\Lambda^2 V$ and that taking this subtlety into account properly should resolve the problem.

We recapitulate the standard field theory situation and the corresponding heterotic string version in *sect. 2*. In *sect. 3* we develop the specific heterotic string theory model we employ and identify in *sect. 4* the puzzling correction term to the expected identity; we also show the appearance of precisely this correction already in an analogous auxiliary problem given by an Euler number computation in a ramified covering (we later argue that because of the coincidence of the complicated inner structure of this correction term in both problems, the original one and the reduced one, our solution of the reduced problem suggests that the essential content of the original problem is already captured); further we give a first discussion of the individual cases $n = 4, 5$ and argue for the necessity to consider also the general case $n \geq 6$. The resolution of the puzzle in the reduced auxiliary problem for this general case is given in *sect. 5*; in the light of this general discussion the cases $n = 4, 5$ are investigated again in greater detail. We conclude in *sect. 6*.

2 Matter multiplets in susy GUT's in field and string theory

For convenience of the reader and to establish some notation we start in assembling some standard facts about matter multiplets in susy GUT's in field and string theory.

2.1 The field theory: matter and Higgs multiplets in susy GUT's

We consider supersymmetric grand unified extensions of the standard model (SM), more precisely theories with one of the GUT groups $SU(5)$ or $SO(10)$ (or E_6).

2.1.1 The standard model

Let us begin by recalling some well-known facts. For the SM itself one has the fermionic matter content

$$SM \text{ fermions} = Q \oplus \bar{u} \oplus \bar{d} \oplus L \oplus \bar{e} \quad (2.1)$$

$$= (\mathbf{3}, \mathbf{2})_{1/3} \oplus (\bar{\mathbf{3}}, \mathbf{1})_{-4/3} \oplus (\bar{\mathbf{3}}, \mathbf{1})_{2/3} \oplus (\mathbf{1}, \mathbf{2})_{-1} \oplus (\mathbf{1}, \mathbf{1})_2 \quad (2.2)$$

(indicating the decomposition under the SM gauge group $SU(3)_c \times SU(2)_{ew} \times U(1)_Y$); one has the anti-particles as well and a net amount of 3 chiral families (besides these unpaired multiplets which appear at low energies the remaining conjugate pairs pair up with high scale masses). Furthermore, in the supersymmetric extension of the standard model (which we consider) the electro-weak Higgs doublet comes as a conjugate pair: $H_u = (\mathbf{1}, \mathbf{2})_1, H_d = (\mathbf{1}, \bar{\mathbf{2}})_{-1}$; the mass term μ sits at the weak scale.

2.1.2 The $SU(5)$ GUT theory

For the GUT group $SU(5)$ the fermionic matter sits in

$$\bar{\mathbf{5}} = L \oplus \bar{d} \quad (2.3)$$

$$\mathbf{10} = Q \oplus \bar{u} \oplus \bar{e} \quad (2.4)$$

(indicating the decomposition under the SM subgroup). The Higgs sits in a $\mathbf{5}$ and under a breaking of $SU(5)$ to the SM gauge group one has to understand the splitting between the electro-weak doublet which has to sit at the weak scale and a remaining color triplet which is superheavy. One gets the conditions (with the net numbers $N_{\mathbf{R}} := n_{\mathbf{R}} - n_{\bar{\mathbf{R}}}$)

$$N_{\bar{\mathbf{5}}} = N_{\mathbf{10}} = 3 \quad (2.5)$$

$$n_{\mathbf{5}} \geq 1 \quad (2.6)$$

If one has a Wilson line breaking to the SM one may pose instead of (2.6) the stronger condition of having just *one* SM Higgs doublet conjugate (vector-like) pair.

2.1.3 The $SO(10)$ GUT theory

For $SO(10)$ the previous fermionic matter content combines together with a right-handed neutrino (which is a singlet under the SM gauge group) to the chiral spinor representation

$$\mathbf{16} = \bar{\mathbf{5}} \oplus \mathbf{10} \oplus \mathbf{1} \quad (2.7)$$

(indicating the decomposition under $SU(5)$) whereas a conjugate pair of $\mathbf{5}$'s of $SU(5)$ (containing a Higgs conjugate pair) combines to a $\mathbf{10}$ of $SO(10)$ (note that this is a real representation). One gets the conditions

$$N_{\mathbf{16}} = 3 \quad (2.8)$$

$$n_{\mathbf{10}} \geq 1 \quad (2.9)$$

If one has a Wilson line breaking to the SM one may pose instead of (2.9) the stronger condition of having just *one* SM Higgs doublet conjugate (vector-like) pair. (Ideally taking the invariant subspace projects out the unwanted colour triplets.)

(In the next step $\mathbf{16}$ and $\mathbf{10}$ would combine with a singlet of $SO(10)$ to the fundamental representation $\mathbf{27}$ of E_6 . This case turns out to be not interesting for us, cf. below.)

2.2 The string theory: heterotic models

A supersymmetric heterotic string model in four dimensions arises as the low energy effective theory from a compactification of the tendimensional heterotic string on a Calabi-Yau threefold X endowed with a polystable holomorphic vector bundle V' . Usually one takes $V' = (V, V_{hid})$ with V a *stable* bundle considered to be embedded in (the visible) E_8 (V_{hid} plays the corresponding role for the second hidden E_8); the commutant of V in E_8 gives the unbroken gauge group in four dimensions. (Though it is not essential we restrict our attention to V .) The μ term will arise as a moduli vev in a cubic interaction $\phi H_u H_d$.

One has in the decomposition of the tendimensional gauge group with respect to $G \times H_V$ (where G is one of the fourdimensional GUT gauge groups $E_6, SO(10)$ and $SU(5)$ and H_V the corresponding structure group $SU(n)$, for $n = 3, 4, 5$, of the bundle V) besides the fourdimensional gauge group $(\mathbf{ad}, \mathbf{1})$ and the singlets $(\mathbf{1}, \mathbf{ad})$ (these two terms are indicated below by "...") the following matter multiplets (with $\mathbf{6} = \Lambda^2 \mathbf{4}$ and $\overline{\mathbf{10}} = \Lambda^2 \overline{\mathbf{5}}$)

$$\mathbf{248} = (\mathbf{16}, \mathbf{4}) \oplus c.c. \oplus (\mathbf{10}, \mathbf{6}) \oplus \dots \quad \text{for } n = 4 \quad (2.10)$$

$$= (\overline{\mathbf{5}}, \overline{\mathbf{10}}) \oplus c.c. \oplus (\mathbf{10}, \overline{\mathbf{5}}) \oplus c.c. \oplus \dots \quad \text{for } n = 5 \quad (2.11)$$

(for $n = 3$ one has just $(\mathbf{27}, \mathbf{3}) \oplus c.c. \oplus \dots$; $\Lambda^2 \mathbf{3}$ is then no new representation of H_V but rather $\overline{\mathbf{3}}$). This leads to the number of matter multiplets as given by²

$$n_{\mathbf{16}} = h^1(V), \quad n_{\mathbf{10}} = h^1(\Lambda^2 V) \quad \text{for } n = 4 \quad (2.12)$$

$$n_{\overline{\mathbf{5}}} = h^1(\Lambda^2 \overline{V}), \quad n_{\mathbf{10}} = h^1(\overline{V}) \quad \text{for } n = 5 \quad (2.13)$$

²all cohomology groups are taken over X unless indicated otherwise

Note that in both cases the number of Higgses is related to a computation of $h^1(\Lambda^2 V)$. This is the rationale for the title of this note and decisive for what we will do in it.

For the net amount of chiral matter one has

$$N_{\mathbf{16}} = -\frac{1}{2}c_3(V) \quad \text{for } n = 4 \quad (2.14)$$

$$N_{\mathbf{10}} = +\frac{1}{2}c_3(V), \quad N_{\bar{\mathbf{5}}} = +\frac{1}{2}c_3(\Lambda^2 V) = +\frac{1}{2}(n-4)c_3(V) \quad \text{for } n = 5 \quad (2.15)$$

Here the case of the self-conjugate (real) $\mathbf{10}$ of $SO(10)$ does not occur, of course.

For example, for $n = 5$ with Wilson line breaking to the SM one may get as low-energy particle spectrum (in a case of absence of exotics, i.e. where $n_{\mathbf{10}}^\pm = 0$ and $n_{\mathbf{5}}^\pm = 0$)

$$n_{\mathbf{10}}^+ = 3, \quad n_{\mathbf{10}}^- = 3, \quad (2.16)$$

$$n_{\mathbf{5}}^+ = 3, \quad n_{\mathbf{5}}^- = 3 + 1 \quad (2.17)$$

$$n_{\mathbf{5}}^+ = 0, \quad n_{\mathbf{5}}^- = 1 \quad (2.18)$$

(where \pm indicates the weight under the \mathbf{Z}_2 Wilson line action). Here (2.16) gives $\bar{u} \oplus \bar{e}$ and Q , (2.17) \bar{d} and L together (" $+$ ") with H_d and (2.18) finally H_u .

3 Specific heterotic models: spectral cover bundles on elliptic Calabi-Yau threefolds

Now we consider a specific class of heterotic models: those where X carries an elliptic fibration $\pi : X \rightarrow B$ with section σ and V is given by the spectral cover construction [1].

3.1 The bundle V itself

So we start with a n -fold ramified cover surface C_V of class³

$$[C_V] = n\sigma + \eta \quad (3.1)$$

(over the embedded base surface⁴ B of class σ) where a line bundle L_V is given with

$$c_1(L_V) = \left[\frac{n\sigma + \eta + c_1}{2} + \lambda(n\sigma - (\eta - nc_1)) \right] \Big|_{C_V} \quad (3.2)$$

³and of equation $w = a_0 + a_2x + a_3y + a_4x^2 + a_5xy$ in Weierstrass coordinates for $n = 5$ (and with $a_5 = 0$ for $n = 4$), cf. [1]; by abuse of notation we will, here and in all similar cases yet to come, denote the cohomology class $[M]$ of a geometric variety M also just by M

⁴The standard examples are the Hirzebruch surfaces \mathbf{F}_k ($k = 0, 1, 2$), the del Pezzo surfaces \mathbf{dP}_k ($k = 0, \dots, 8$) and the Enriques surface.

(where $c_1 := c_1(B)$, $\eta \in H^2(B, \mathbf{Z})$ and all pull-backs by π here and below are suppressed). An important role is played by the intersection curve $A_V := C_V \cap \sigma$ which can be considered to lie either in C_V or in $\sigma = \sigma(B) = B$: one knows that the first cohomology of the bundle V is localised along the curve in B where one of the n line bundle summands (into which V decomposes when restricted to a fibre) becomes trivial, i.e. where one of the n fibre points of the cover surface C_V meets σ ; this leads to the curve (with corresponding cohomology class³)

$$A_V = C_V \cap B \quad (3.3)$$

$$[A_V] = \eta - nc_1 \quad (3.4)$$

3.1.1 Consideration of the ramification locus

Let us denote by r_V the ramification locus (above in C_V) of $\pi : C_V \rightarrow B$ (as usual r_V will denote also the cohomology class). From $K_{C_V} = \pi_{C_V}^* K_B + r_V$ one finds

$$r_V = n\sigma + \eta + c_1|_{C_V} \quad (3.5)$$

$$= n[2\eta - (n-1)c_1]\sigma + \eta(\eta + c_1) \quad (3.6)$$

This is the class of the curve on C_V of ramification points (above), i.e. of points where in a fibre some points coincide: $p_i = p_j$ ($i \neq j$). For the class of self-mirror points among these doubly counting points of C_V (if $p_i + p_j = 0$, or equivalently $p_i = \tau p_j$ under the standard involution τ [1], we call the pair (p_i, p_j) τ -conjugate or a mirror pair) one finds

$$r_V \cdot \sigma_{II} = 4\eta^2 - 2(n-2)\eta c_1 + n(n-1)c_1^2 \quad (3.7)$$

(where $\sigma_{II} = \sigma + \sigma_2$ denotes the fourfold section of two-torsion points and σ_2 is the trisection of *strict* two-torsion points; the two components of σ_{II} being disjoint one gets for σ_2 the class $3(\sigma + c_1)$).

3.2 The derived bundle $\Lambda^2 V$

The bundle $\Lambda^2 V$ is again stable and spectral [3] (we assume $n \geq 3$ from now on), its cover surface $C_{\Lambda^2 V}$ has⁵ class

$$[C_{\Lambda^2 V}] = \frac{n(n-1)}{2}\sigma + (n-2)\eta \quad (3.8)$$

⁵this follows with $c_2(\Lambda^2 V) = (n-2)c_2(V)$ from the general representation of $c_2(U)$ as $\eta_U \sigma + \omega$ for any spectral bundle U with class $C_U = rk(U)\sigma + \eta_U$ of its cover surface C_U

Similarly as for the original bundle V one realises, when considering $\Lambda^2 V$ just as another spectral bundle in itself, that its first cohomology is localised along the curve⁶

$$A_{\Lambda^2 V} = C_{\Lambda^2 V} \cap B \quad (3.9)$$

$$[A_{\Lambda^2 V}] = (n-2)\eta - \frac{n(n-1)}{2}c_1 \quad (3.10)$$

We assume from now on that c_1 is effective (thus excluding among the standard bases just the Enriques surface, cf. footn. 4) such that with A_V of class $\eta - nc_1$ also $(n-2)\eta - n(n-2)c_1$ is effective and thus also (3.9).

One can also, however, consider this locus from the perspective of the original bundle V , from which $\Lambda^2 V$ is derived, and *its* spectral cover surface C_V . Then one finds that $H^1(\Lambda^2 V)$ is localised along the curve of points b in B where in the corresponding fibre $C_b = \pi^{-1}(b)$ two *generically different* points p_i and p_j of C_V fulfil $p_i + p_j = 0$ (for, if p_i are the points of C_V on C_b , then the $p_i + p_j$ with $i < j$ are the points of $C_{\Lambda^2 V}$ there) or, equivalently, are conjugated under the standard involution τ . Thus $A_{\Lambda^2 V}$ has as double cover the *non-trivial* divisor component $\tilde{A}_{\Lambda^2 V}$ in $C_V \cap \tau C_V$, the other ('trivial') components being given by $C_V \cap \sigma_{II}$. Now, similarly as the surface C_V , an n -fold ramified cover of B , does not just have $n\sigma$ as its cohomology class but rather $n\sigma + \eta$, here the curve $\tilde{A}_{\Lambda^2 V}$ has not just $2A_{\Lambda^2 V}\sigma$ as cohomology class; rather one gets, by decomposing the intersection of C_V with its conjugate (which has the same cohomology class),

$$\tilde{A}_{\Lambda^2 V} = [C_V - \sigma_{II}]|_{C_V} = [(n-4)\sigma + \eta - 3c_1]|_{C_V} \quad (3.11)$$

$$= 2A_{\Lambda^2 V}\sigma + \eta(\eta - 3c_1) \quad (3.12)$$

(where the second line is to be read in X). From the ramified double covering $\pi_C : \tilde{A}_{\Lambda^2 V} \rightarrow A_{\Lambda^2 V}$ one gets (as relation of degrees)

$$\frac{1}{2}c_1(T\tilde{A}_{\Lambda^2 V})|_{\tilde{A}_{\Lambda^2 V}} = c_1(TA_V)|_{A_V} - \frac{1}{2}\sigma_{II} \cdot \tilde{A}_{\Lambda^2 V} \quad (3.13)$$

(after multiplying the Euler number relation by $1/2$).

⁶Note that the localization curve A_V for V (where the matter computations take place) can be 'turned off' discretely by an appropriate choice of η : just take $\eta - nc_1 = 0$ (which is of course still effective); so for this choice $c_3(V)$ should vanish - as it does indeed. If one tries to apply the same logic to the localization curve $A_{\Lambda^2 V}$ of $\Lambda^2 V$, however, one realises that the corresponding choice $\eta = \frac{n(n-1)}{2(n-2)}c_1$ (let us assume this would be an integral class, say for $n = 3$ or 4) would violate the demand that $\eta - nc_1$ should be effective; thus the analogous procedure is not legitimate there, which is good as $c_3(\Lambda^2 V)$, being proportional to $c_3(V)$, does *not* have a cohomological factor $[A_{\Lambda^2 V}]$ but still just $[A_V]$. (This remark is relevant for $n = 4$; for $n = 3$ the two choices of η , and already $[A_V]$ and $[A_{\Lambda^2 V}]$ are identical and so no contradiction arises there (of course, for $n = 3$ already $\Lambda^2 V$ itself is just \bar{V}).)

For the (degree of the) Chern class of the line bundle $L_{\Lambda^2 V}$, when restricted to the corresponding intersection curve, one gets according to [3]

$$c_1(L_{\Lambda^2 V})|_{A_{\Lambda^2 V}} = \left[c_1(L_V) - \frac{1}{2}\sigma_{II} \right] \cdot \tilde{A}_{\Lambda^2 V} \quad (3.14)$$

4 The puzzle

Application of the Leray spectral sequence to the elliptic fibration gives $H^1(X, U) \cong H^0(A_U, (L_U \otimes K_B)|_{A_U})$ where U is either V or $\Lambda^2 V$ (or $\overline{V}, \Lambda^2 \overline{V}$). This gives the estimate²

$$\begin{aligned} h^1(U) &\geq \deg((L_U \otimes K_B)|_{A_U}) + 1 - g_{A_U} \\ &= \left[c_1(L_U) - \frac{A_U + c_1}{2} \right] \cdot A_U \end{aligned} \quad (4.1)$$

The one term which is missing in our computation here (which uses the Riemann-Roch formula just as an estimate instead of an equation) is precisely the 'other' term describing the anti-multiplet, which combines (subtractively) with the estimated term to give the *net* amount; this other term is

$$h^1(\overline{U}) = h^0(A_U, (L_U \otimes K_B)^*|_{A_U} \otimes K_{A_U}) \quad (4.2)$$

The final identification just mentioned holds for $U = V$ in general, therefore one expects⁷ it to hold also for $U = \Lambda^2 V$. Here one assumes that the curve $A_{\Lambda^2 V}$ is smooth, reduced and irreducible. According to [3] one has indeed that *generically* (although perhaps not in specific phenomenologically relevant cases) the curve $A_{\Lambda^2 V}$ is smooth and irreducible⁸. One should note that when the authors of [3] make computations in a *specific* model they do not use this assumption.

Therefore one expects

$$h^1(U) - h^1(\overline{U}) = \deg((L_U \otimes K_B)|_{A_U}) + \frac{1}{2}e_{A_U} \quad (4.3)$$

$$= \left[c_1(L_U) - \frac{A_U + c_1}{2} \right] \cdot A_U \quad (4.4)$$

Evaluating this for $U = V$ gives $-\lambda\eta(\eta - nc_1)$, thus giving, according to [2], indeed the correct result $-\frac{1}{2}c_3(V)$.

⁷Although there is a certain difference between $U = V$ and $U = \Lambda^2 V$ - the latter has a spectral line bundle $L_{\Lambda^2 V}$ which is not the restriction of a line bundle on X to $C_{\Lambda^2 V}$ - this is not relevant for the assumptions going in (4.2) and thus not relevant for the point in question.

⁸with "irreducible" possibly intended to mean reduced and irreducible as assumptions for the applicability of the ordinary Riemann-Roch formula are at stake there

When doing the corresponding computation for $U = \Lambda^2 V$ one meets a surprise, however: one gets (with (3.14)) for $h^1(U) - h^1(\bar{U})$, instead of having just the expected term $-\frac{1}{2}c_3(\Lambda^2 V) = -(n-4)\frac{1}{2}c_3(V) = -(n-4)\lambda\eta(\eta - nc_1)$, in addition also a correction term

$$\deg((L_{\Lambda^2 V} \otimes K_B)|_{A_{\Lambda^2 V}}) + \frac{1}{2}e_{A_{\Lambda^2 V}} = \Delta_n - (n-4)\lambda\eta(\eta - nc_1) \quad (4.5)$$

for which one has the following expression

$$\Delta_n = -\frac{n-3}{2} \left[(n-4)\eta^2 - (n^2 - 3n - 2)\eta c_1 + n(n-1)\frac{n-2}{4}c_1^2 \right] \quad (4.6)$$

$$= \begin{cases} (\eta - 3c_1)c_1 & \text{for } n = 4 \\ -(\eta - 3c_1)(\eta - 5c_1) & \text{for } n = 5 \end{cases} \quad (4.7)$$

(Δ_3 vanishes but there we had not to consider any $\Lambda^2 V$; we assume $n \geq 4$ from now on).

4.1 An analogous, reduced problem

To locate the point where an error might have come in we switch to an auxiliary problem: as a mismatch does occur in the mentioned expressions let us check whether at least the simpler Euler number comparison in (3.13) comes out correctly. One finds with⁹

$$K_{\tilde{A}_{\Lambda^2 V}} = 6[n-2]\eta^2 - 2[3n^2 - 3n - 4]\eta c_1 + n(n-1)[2n-1]c_1^2 \quad (4.8)$$

$$K_{A_{\Lambda^2 V}} = [n-2]^2\eta^2 - [(n-2)(n^2 - n + 1)]\eta c_1 + n(n-1)\left[\frac{n^2 - n + 2}{4}\right]c_1^2 \quad (4.9)$$

$$\deg Br = 4\eta^2 - 2[n+4]\eta c_1 + n(n-1)c_1^2 \quad (4.10)$$

that one has, surprisingly, the relation again with a correction term

$$\frac{1}{2}e_{\tilde{A}_{\Lambda^2 V}} = e_{A_{\Lambda^2 V}} - \frac{1}{2}\deg Br - 2\Delta_n \quad (4.11)$$

As the same correction term (up to a constant factor) occurs here just in an Euler number computation we will investigate now more closely its structure, especially in connection to the ramification structure of double covering $\tilde{A}_{\Lambda^2 V} \rightarrow A_{\Lambda^2 V}$.

The philosophy we follow from now on will be that the appearance of precisely the same correction term already in the analogous auxiliary problem (given by an Euler number computation in a ramified covering) does not just represent an accidental coincidence (the rationale for this philosophy consists in the complicated inner structure of this same correction term for both problems, the original one and the reduced one). Therefore, we argue, our later explanation and complete solution of the reduced problem suggests that the source and essential content of the original problem is already captured.

⁹where the degree $\deg Br$ of the branching locus below in $A_{\Lambda^2 V} \subset B$ can also be computed above in $\tilde{A}_{\Lambda^2 V} \subset C_V$, namely from the ramification locus as $\tilde{A}_{\Lambda^2 V} \cdot \sigma_{II}$

4.2 The sign of the correction term

For the physical application the cases $n = 4$ and 5 are, of course, the ones of greatest interest. However, the 'puzzle' exists already on a purely mathematical level, and so the contradiction has to be solved for all $n \geq 6$ as well (for $n = 3$ the correction vanishes).

Let us investigate first the sign of the correction term. One of the assumptions in the spectral cover set-up is that $e_n := \eta - nc_1$, being the class of A_V in B , is (the class of an) effective (divisor). Thus we get

$$\Delta_n = \begin{cases} (e_4 + c_1)c_1 & \text{for } n = 4 \\ -(e_5 + 2c_1)e_5 & \text{for } n = 5 \end{cases} \quad (4.12)$$

Let us assume from now on that c_1 is ample. This excludes among the bases \mathbf{F}_k (with $k = 0, 1, 2$), \mathbf{dP}_k (with $k = 0, \dots, 8$) and Enriques (which was already excluded) just the latter and \mathbf{F}_2 . This gives $\Delta_4 > 0$. For $n = 5$ one may wish to assume that e_5 is not only effective but even ample (if C_V is ample then $e_n = [A_V]$ will be ample, too): then one gets $\Delta_5 < 0$.

As explained above the correction issue has also to be explained for all $n \geq 6$. Now, for $n = 6$ one finds that with $\Delta_6 = 3\Delta_5$ the same reasoning as in the case $n = 5$ applies. One could now compute the higher Δ_n individually and consider them case by case; one finds, for example, $\Delta_8 = -5(e_8 + \eta + c_1)(e_8 + 2c_1)$. In general one has

$$\Delta_n = -\frac{n-3}{2} \left[(n-4)e_n^2 + (n^2 - 5n + 2)e_nc_1 + n\frac{(n-2)(n-5)}{4}c_1^2 \right] \quad (4.13)$$

One learns that for e_n ample the term in big brackets here is always positive for $n \geq 5$. We will assume from now on that $e_n = [A_V]$ is ample. We find that the expression $\delta_n := -2\Delta_n$ (the rationale for this definition will become clear below) is always positive for $n \geq 5$.

4.3 A preliminary look at the cases $n = 4$ and $n = 5$

So, one of the assumptions going in the derivation of (3.14) (or possibly of (4.2)) must be wrong. Clearly all the computations above were done only for the case where the curve $A_{\Lambda^2 V}$ is as 'good' as possible; now, the assumption of [3] was that in the *generic* case (though not in one or another special case which may be of phenomenological interest) this curve is as 'good' as one could wish, namely smooth, reduced and irreducible, *and* that the map $\pi : \tilde{A}_{\Lambda^2 V} \rightarrow A_{\Lambda^2 V}$ is an ordinary ramified double covering.

This assumption turns out to be wrong, however: the curve and the covering in question are 'special' already generically. We will now show that already generically the curve $A_{\Lambda^2 V}$ is non-reduced for $n = 4$ and the covering has points with special behaviour (this will be specified later) for $n = 5$. (For the following recall that in the group law on the elliptic fibre the n points of C on a fibre add up to zero: $\sum_{i=1}^n p_i = 0$.)

4.3.1 The case $n = 4$

For $n = 4$ one has the cohomology classes $\tilde{A}_{\Lambda^2 V} = (4\sigma + \eta)(\eta - 3c_1)$ and $A_{\Lambda^2 V} = 2(\eta - 3c_1)$ and sees that if we are on a C_V -fibre over a point of $A_{\Lambda^2 V} \subset B$ where two points p_1 and p_2 are τ -conjugate to each other, in other words where $p_1 + p_2 = 0$, then one will also have $p_3 + p_4 = 0$, i.e. each point of the curve 'counts twice'. The curve turns out to be two times its reduced version: the cohomology class of $A_{\Lambda^2 V}$ shows already that this is indeed possible and consideration [1] of the τ -action in coordinates shows that the reduced curve is actually given by $a_3 = 0$ (cf. footn. 3) which has just the correct class $\eta - 3c_1$. (The locus of 'special' points (which can cause deviations) - here the curve as a whole - is insofar in harmony with the correction $\Delta_4 = (\eta - 3c_1)c_1$ we found as the latter is 'proportional' to (the cohomology class of) the former: $\Delta_4 = \frac{1}{2}A_{\Lambda^2 V}c_1$; but one can not 'turn off' the special locus by the choice $\eta = 3c_1$ which violates the effectivity of e_4).

4.3.2 The case $n = 5$

For $n = 5$, where one has $A_{\Lambda^2 V} = 3\eta - 10c_1$, one gets similarly in a C_V -fibre over a point of our curve $A_{\Lambda^2 V} \subset B$ that $p_3 + p_4 + p_5 = 0$; so here the 'doubly counting' points¹⁰ in the base curve $A_{\Lambda^2 V}$ (of the covering $\pi : \tilde{A}_{\Lambda^2 V} \rightarrow A_{\Lambda^2 V}$) are just the points where, say, p_5 is zero. This leads to an investigation of the intersection locus consisting of the points where our curve meets the other curve¹¹ A_V . The number of these points is $\nu_5 = (3\eta - 10c_1) \cdot (\eta - 5c_1)$. The locus $A_{\Lambda^2 V} \cap A_V$ we investigate here (as the possible source of the deviation) is not yet precisely the source of the correction Δ_5 as the latter is not proportional¹² to ν_5 . This discrepancy is easily explained, we prefer, however, to delay this consideration to a later section and will give first a systematic study of all relevant special loci in the problem in general.

¹⁰where a second pair of points $\{p_k, p_l\}$ (where $k \neq l$) with $p_k + p_l = 0$ exists besides the pair $\{p_1, p_2\}$ (so one needs to demand here $\{p_k, p_l\} \neq \{p_1, p_2\}$), cf. remark in the next paragraph

¹¹where one of the fibre points of C_V is zero; it has class $\eta - 5c_1$ which is the class of the vanishing locus of a_5 (cf. footn. 3), i.e. of the locus of $A_V = C_V \cap \sigma$

¹²one can discretely tune to zero ν_5 (by a suitable choice of $\eta - 5c_1$, say $15b + 6f$ on \mathbf{F}_1) while having nevertheless still $\Delta_5 \neq 0$ (actually -9 in the mentioned example)

We remark that the proper interpretation of the phrase 'doubly counting' used above will also be investigated further below in sect. 5: for the moment 'doubly counting' refers just to points $b \in B$ which have points p_i, p_j, p_k, p_l in the fibre $C_b := \pi^{-1}(b)$ of C_V over b with $p_i + p_j = 0$ ($i \neq j$) and $p_k + p_l = 0$ ($k \neq l$) where the index sets $\{i, j\}$ and $\{k, l\}$ are different: $\{i, j\} \neq \{k, l\}$. It will be important to distinguish the latter condition from the stronger requirement that one has even $\{i, j\} \cap \{k, l\} = \emptyset$. That the latter, stronger condition is also fulfilled was clear in the case of $n = 4$ when the existence of a specific second pair was derived; to have the same also in the argument we used for $n = 5$ one has to make sure that the point playing the role of p_5 above was not already among the pair consisting of the points p_1 and p_2 (which led us to a generic point of the curve $A_{\Lambda^2 V}$ in the first instance); the latter possibility (that p_5 is either p_1 or p_2 , where then in fact $p_1 = 0 = p_2$) - and its impact of splitting different effects in the geometry which contribute to the correction - will be considered in detail below in sect. 5.

So we have found that the relevant curve $A_{\Lambda^2 V}$ has special loci (even generically) for $n = 4$ (where the curve is nonreduced) and $n = 5$ (where 'doubly counting' points exist whose precise interpretation has yet to be investigated); one should find that, if one takes into account appropriately in all computations the special nature of the curve $A_{\Lambda^2 V}$ and the special nature of the covering near the special points, the correction term vanishes.

4.3.3 Necessity to consider also the more general case $n \geq 6$

The correction term Δ_n exists also, however, for $n \geq 6$. Apart from the arguments given above for the cases $n = 4$ and 5 one should, for reasons of mathematical consistency, therefore also find points of $A_{\Lambda^2 V}$ which are 'special' for the covering $\pi : \tilde{A}_{\Lambda^2 V} \rightarrow A_{\Lambda^2 V}$ for $n \geq 6$ (and one should find furthermore that, taking these properly into account in all computations, the correction in the end vanishes). These considerations show that the arguments used so far are in any case incomplete. For the reasoning we have used up to now is no longer sufficient to treat these cases: in a fibre over a point of our curve $A_{\Lambda^2 V}$ we find only that $p_3 + p_4 + \dots + p_{n-1} + p_n = 0$ and it is not immediately clear whether a further pair $\{p_k, p_l\}$ (where $k \neq l$) with $p_k + p_l = 0$ (and $\{1, 2\} \cap \{k, l\} = \emptyset$) is enforced.

These preliminary considerations will be refined for all $n \geq 6$ in sect. 5. There we will also carefully take into account the difference of having only $\{1, 2\} \neq \{k, l\}$ or the stronger $\{1, 2\} \cap \{k, l\} = \emptyset$. Furthermore the relation to the branching phenomenon, where in a pair p_i, p_j (where $i \neq j$) with $p_i + p_j = 0$ one has actually $p_i = p_j$, will be considered. Thereby we will be able to solve the reduced problem. After having described this we will go back in the final subsection 5.3 again to the cases $n = 4$ and $n = 5$.

5 The resolution

To resolve the discrepancies above let us have now a closer look at the fibration structure of the ramified two-fold covering $\pi : \tilde{A}_{\Lambda^2 V} \rightarrow A_{\Lambda^2 V}$.

5.1 Consideration of the ramification locus

Above the branching of the map $\pi : \tilde{A}_{\Lambda^2 V} \rightarrow A_{\Lambda^2 V}$ was computed as follows (we denote, following [3], the ramification divisor above by R and the branching divisor below by Br)

$$\deg Br = \deg R = \tilde{A}_{\Lambda^2 V} \cdot \sigma_{II} \quad (5.1)$$

$$= 4\eta^2 - 2(n+4)\eta c_1 + n(n-1)c_1^2 \quad (5.2)$$

The ramification locus R (above) is the locus where two generically different (this is the definition of $\tilde{A}_{\Lambda^2 V}$) fibre points p_i, p_j ($i \neq j$) with $p_i = \tau p_j$ (let us call such a pair of fibre points henceforth a mirror pair for short) come together. Clearly the limit point (the coalescing point) of such a mirror pair is a self-mirror point, i.e. a 2-torsion point, i.e. an element of $C \cap \sigma_{II}$; so one has (here the first inclusion is an identity¹³)

$$R \subset \tilde{A}_{\Lambda^2 V} \cap \sigma_{II} \subset C_V \cap \sigma_{II} \quad (5.3)$$

Points of the set $C_V \cap \sigma_{II}$ which are such limits, i.e. elements of R , are in particular doubly counting points of C_V (but this is not sufficient); so one has

$$R \subset r_V \cap \sigma_{II} \subset C_V \cap \sigma_{II} \quad (5.4)$$

Here the first inclusion is however not an equality: although a two-torsion point p_* (in a fibre C_b of C_V) which is actually also doubly counting (in C_b) is the limit of different fibre points in neighbouring fibers which approach the fibre C_b (containing p_*) these neighbouring point pairs do not have to be mirror pairs. Therefore one expects $\deg R < r_V \cdot \sigma_{II}$.

Actually one has

$$r_V \cdot \sigma_{II} = \tilde{A}_{\Lambda^2 V} \cdot \sigma_{II} + 12\eta c_1 \quad (5.5)$$

and thus one finds indeed just the sort of relation one expects here (we mean the relation $\deg R = \tilde{A}_{\Lambda^2 V} \cdot \sigma_{II} < r_V \cdot \sigma_{II}$) because we assume that c_1 is ample so that $\eta c_1 > 0$ (as

¹³the limit point of a coalescing mirror pair is a self-mirror point; conversely, a self-mirror point p_* on a branch of the curve (of mirror pairs) $\tilde{A}_{\Lambda^2 V}$ is a point where the two branches of $\tilde{A}_{\Lambda^2 V}$ meet, as the mirror partner of p_* is again p_* (such a point could also be a double point if $\tilde{A}_{\Lambda^2 V}$ would not be smooth)

$\eta - nc_1$ has to be effective). More precisely, one finds that the σ_2 -term of $\sigma_{II} = \sigma + \sigma_2$ causes the difference as one has (a relation which itself is not a priori obvious¹⁴)

$$\tilde{A}_{\Lambda^2 V} \cdot \sigma = A_V \cdot A'_V = r_V \cdot \sigma \quad (5.6)$$

where we use the notation $A'_V := \eta - (n-1)c_1$ (as cohomology class), a class 'derived' from the class $A_V = \eta - nc_1$.

Remark: The considerations above show also that inside the curve A_V , which consists of points of $b \in B$ where a point $p_i = 0$ in the fibre C_b of C_V exists, there exist a number of $r_V \cdot \sigma = A_V \cdot A'_V$ points where actually two points of $p_i = 0 = p_j$ ($i \neq j$) exist in C_b . Below in (5.11) we will extrapolate this to $A_{\Lambda^2 V}$ and will give a detailed interpretation.

One finds now for (the class of) the ramification locus $r_{\Lambda^2 V}$ of $\pi : C_{\Lambda^2 V} \rightarrow B$

$$r_{\Lambda^2 V} = n(n-1) \left[(n-2)\eta - \frac{n^2 - n - 2}{4} c_1 \right] \sigma + (n-2)\eta \left[(n-2)\eta + c_1 \right] \quad (5.7)$$

One has, in analogy to (5.6), here the following relation (again with $A'_{\Lambda^2 V} := A_{\Lambda^2 V} + c_1$)

$$r_{\Lambda^2 V} \cdot \sigma = A_{\Lambda^2 V} \cdot A'_{\Lambda^2 V} \quad (5.8)$$

Let us distinguish in a more detailed manner various subsets which are relevant in the ramification of the covering $\pi : \tilde{A}_{\Lambda^2 V} \rightarrow A_{\Lambda^2 V}$ (for the meaning of the indices k, j cf. below)

$$\pi(\tilde{A}_{\Lambda^2 V} \cap \sigma_{II}) \subset \pi(\tilde{A}_{\Lambda^2 V} \cap r_V) \quad (k = j \text{ allowed}) \quad (5.9)$$

$$\left(\pi(\tilde{A}_{\Lambda^2 V} \cap r_V) \setminus \pi(\tilde{A}_{\Lambda^2 V} \cap \sigma_{II}) \right) \subset r_{\Lambda^2 V} \cap \sigma \subset A_{\Lambda^2 V} \quad (k \neq j) \quad (5.10)$$

where the different subsets have the following interpretation (with $p_i \in C_b := C_V \cap \pi^{-1}(b)$)

$$r_{\Lambda^2 V} \cap \sigma = \{b \in B \mid p_i + p_j = 0 = p_k + p_l \text{ where } i \neq j, k \neq l \text{ and } \{i, j\} \neq \{k, l\}\} \quad (5.11)$$

$$\begin{aligned} \pi(\tilde{A}_{\Lambda^2 V} \cap r_V) &= \{b \in B \mid p_i + p_j = 0 = p_k + p_j \text{ where } i \neq j \text{ and } k \neq i\} \\ &= \{b \in B \mid p_i + p_j = 0 \text{ and } p_i = p_k \text{ where } i \neq j \text{ and } k \neq i\} \end{aligned} \quad (5.12)$$

$$\pi(\tilde{A}_{\Lambda^2 V} \cap \sigma_{II}) = \{b \in B \mid p_i + p_j = 0 \text{ and } p_i = p_j \text{ where } i \neq j\} \quad (5.13)$$

where in (5.12), (5.13) the mirror pair $\{p_i, p_j\}$ is assumed to occur in a family of mirror pairs consisting generically of different¹⁵ points. In (5.12) the interpretation of r_V is that

¹⁴as $\tilde{A}_{\Lambda^2 V} \cdot \sigma$ concerns the case where the coalescing point of a *mirror* pair is zero whereas in $r_V \cdot \sigma$ the coalescing point of a pair (an arbitrary pair of fibre points coming together in a special fibre) is zero.

¹⁵the feature of the curve $\tilde{A}_{\Lambda^2 V}$ distinguishing it from the other components $\sigma|_C$ and $\sigma_2|_C$ of $\tau C|_C$

one has $k \neq i$ although it is still possible that $k = j$; the latter case leads to the first inclusion (5.9) (the condition in (5.13) describes the ramification points of the covering $\pi : \tilde{A}_{\Lambda^2 V} \rightarrow A_{\Lambda^2 V}$). By contrast the case $k \neq j$ leads to the second inclusion (5.10).

Let us also give the cardinalities of the sets (5.11) - (5.13)

$$r_{\Lambda^2 V} \cdot \sigma = [n^2 - 4n + 4]\eta^2 - [n^3 - 3n^2 + n + 2]\eta c_1 + n(n-1)\frac{n^2 - n - 2}{4}c_1^2 \quad (5.14)$$

$$\tilde{A}_{\Lambda^2 V} \cdot r_V = [3n - 4]\eta^2 - [3n^2 - 4n + 4]\eta c_1 + n(n-1)[n-1]c_1^2 \quad (5.15)$$

$$\tilde{A}_{\Lambda^2 V} \cdot \sigma_{II} = 4\eta^2 - [2n + 8]\eta c_1 + n(n-1)c_1^2 \quad (5.16)$$

(as on points sets π is generically one-to-one the cardinalities can be computed upstairs)¹⁶.

In view of the inclusions (5.9), (5.10) also the following sets are interesting (with the same assumption for the occurring mirror pair $\{p_i, p_j\}$ as above)

$$\begin{aligned} \pi(\tilde{A}_{\Lambda^2 V} \cap r_V) \setminus \pi(\tilde{A}_{\Lambda^2 V} \cap \sigma_{II}) &= \{b \in B \mid p_i + p_j = 0 \text{ and } p_i = p_k \\ &\quad \text{where } i \neq j \text{ and } k \neq i, j\} \end{aligned} \quad (5.17)$$

$$\begin{aligned} (r_{\Lambda^2 V} \cap \sigma) \setminus (\pi(\tilde{A}_{\Lambda^2 V} \cap r_V) \setminus \pi(\tilde{A}_{\Lambda^2 V} \cap \sigma_{II})) &= \{b \in B \mid p_i + p_j = 0 = p_k + p_l \\ &\quad \text{where } i \neq j, k \neq l \\ &\quad \text{and } \{i, j\} \cap \{k, l\} = \emptyset\} \end{aligned} \quad (5.18)$$

(i.e. in both cases all indices are different). Let us also introduce their cardinalities

$$\epsilon := \#[\pi(\tilde{A}_{\Lambda^2 V} \cap r_V) \setminus \pi(\tilde{A}_{\Lambda^2 V} \cap \sigma_{II})] \quad (5.19)$$

$$\delta := \#[(r_{\Lambda^2 V} \cap \sigma) \setminus (\pi(\tilde{A}_{\Lambda^2 V} \cap r_V) \setminus \pi(\tilde{A}_{\Lambda^2 V} \cap \sigma_{II}))] \quad (5.20)$$

ϵ and δ cover the cases $\{i, j\} \cap \{k, l\} \neq \emptyset$ and $= \emptyset$, resp. (always $\{i, j\} \neq \{k, l\}$). One has

$$\begin{aligned} \epsilon &= \tilde{A}_{\Lambda^2 V} \cdot r_V - \tilde{A}_{\Lambda^2 V} \cdot \sigma_{II} \\ &= [3n - 8]\eta^2 - [3n^2 - 6n - 4]\eta c_1 + n(n-1)[n-2]c_1^2 \end{aligned} \quad (5.21)$$

$$\begin{aligned} \delta &= r_{\Lambda^2 V} \cdot \sigma - (\tilde{A}_{\Lambda^2 V} \cdot r_V - \tilde{A}_{\Lambda^2 V} \cdot \sigma_{II}) \\ &= [n^2 - 7n + 12]\eta^2 - [n^3 - 6n^2 + 7n + 6]\eta c_1 + n(n-1)\frac{n^2 - 5n + 6}{4}c_1^2 \end{aligned} \quad (5.22)$$

From the interpretation (5.18) one learns that the set of points in the curve $A_{\Lambda^2 V}$ whose cardinality δ is computed in (5.22) is the set of points where the generically two-fold covering $\tilde{A}_{\Lambda^2 V} \rightarrow A_{\Lambda^2 V}$ is actually a four-fold one, consisting of the points p_i, p_j, p_k, p_l : these

¹⁶we will learn later, in sect. 5.3.1, that for this statement to be true one has to demand $n > 4$

points are all distinct because p_i and p_j (and similarly p_k and p_l) are distinct (because the point locus $\pi(\tilde{A}_{\Lambda^2 V} \cap \sigma_{II})$ over which they can coincide is generically distinct from the one in question here) and none of the points p_k, p_l can equal any of the other points p_i, p_j because the set $(\tilde{A}_{\Lambda^2 V} \cap r_V) \setminus (\tilde{A}_{\Lambda^2 V} \cap \sigma_{II})$ was taken out (all indices here are different).

Here one has to make sure that these second special pairs $\{p_k, p_l\}$ with $p_k + p_l = 0$ (besides the first, ordinary pair $\{p_i, p_j\}$ with $p_i + p_j = 0$) above in C_V are actually elements of the *curve* $\tilde{A}_{\Lambda^2 V}$; this means that they too are part of a continuous family of such mirror pairs and not just isolated point pairs with the mirror property. For note that $\tilde{A}_{\Lambda^2 V}$ was *defined* as a divisor by

$$\tau C_V|_{C_V} = \sigma_{II}|_{C_V} + \tilde{A}_{\Lambda^2 V} \quad (5.23)$$

(i.e. essentially as a *curve*). It was *not defined*, (a priori) differing, as the *set* given by

$$\tau C_V \cap C_V \stackrel{?}{=} (\sigma_{II} \cap C_V) \cup \tilde{A}_{\Lambda^2 V} \quad (5.24)$$

When one just considers the covering $\pi : \tilde{A}_{\Lambda^2 V} \rightarrow A_{\Lambda^2 V}$ one learns only that, going away continuously in $A_{\Lambda^2 V}$ from a special point, at least one of the two mirror pairs $\{p_i, p_j\}$ and $\{p_k, p_l\}$ above must have also a continuous continuation which then constitutes a local part of the covering curve; the other mirror pair might, a priori, be an isolated pair.

The existence of the second pair was implied, however, by consideration of the set $r_{\Lambda^2 V} \cap \sigma$. The ramification curve $r_{\Lambda^2 V}$ is the locus where two local surface branches of the covering $\pi_{\Lambda^2 V} : C_{\Lambda^2 V} \rightarrow B$ meet. These local surface branches meet also in X the embedded surface B , i.e. $\sigma(B)$.¹⁷ These intersections constitute local parts of the curve $A_{\Lambda^2 V} = C_{\Lambda^2 V} \cap \sigma$. These local parts give, here in the base curve $A_{\Lambda^2 V}$, the respective continuations (away from the special point in $r_{\Lambda^2 V} \cap \sigma$) which correspond to the searched for continuations of the two mirror pairs above.

One still has to clarify whether these special points in the base curve $A_{\Lambda^2 V}$ of the fibration are singularities (double points) or smooth points like ramification points¹⁸ (in some *other* covering) where different branches meet. According to [3], sect. 7.5, the generic curve $A_{\Lambda^2 V}$ is smooth, so it has to be the latter option. This property of a *smooth* meeting of local branches of $A_{\Lambda^2 V} = C_{\Lambda^2 V} \cap \sigma$ in a special point can be considered (for yet another heuristic argument cf. below) as being inherited from the smooth meeting of local branches of $C_{\Lambda^2 V}$ in the ramification curve $r_{\Lambda^2 V}$ (which generically also is not a curve of double points).

¹⁷we usually identify the base B , the section $\sigma : B \rightarrow X$, its image in X and its cohomology class

¹⁸which in a dimensionally reduced, real picture can not be distinguished from double points: recall the standard picture for a representation of an elliptic curve as fourfold ramified double cover of \mathbf{P}^1

Let us recapitulate how the described non-generic behaviour of the fibre of the map $\tilde{A}_{\Lambda^2 V} \rightarrow A_{\Lambda^2 V}$ is possible. Note that when at such a special fibre the curve $\tilde{A}_{\Lambda^2 V} \subset C_V$ upstairs is projected down to the curve $A_{\Lambda^2 V} = C_{\Lambda^2 V} \cap B \subset B$ then two local branches of $A_{\Lambda^2 V}$ meet there (projections of local parts of $\tilde{A}_{\Lambda^2 V}$ representing continuations of the mirror pairs $\{p_i, p_j\}$ and $\{p_k, p_l\}$). However, $A_{\Lambda^2 V}$ does not have double points there: the generic $A_{\Lambda^2 V}$ is smooth [3]. We now give a second heuristic explanation of the latter fact.

One may compare the case $A_V = C_V \cap B$ which also does *not* have double points at the set of special points $r_V \cap \sigma$ (which are also 'doubly counting' as one has there fibre points with $p_i = 0 = p_j$ ($i \neq j$)). The surface C_V is completely 'general', i.e. it is just a generic member of the *full* linear system $|C_V|$, and actually $A_V = C_V \cap B$ is also completely generic in its linear system $|A_V|$ in B and so the generic member is smooth; at the points of $r_V \cap \sigma$ two local branches of A_V meet in a *non-singular* point like in a ramification point. For $C_{\Lambda^2 V}$ the situation is different at first: this surface is not a generic member of its linear system, rather it arises fibrewise by adding up (in the group law on the fibre) the corresponding fibre points of C_V (which themselves were generic); however, when now intersecting with σ (i.e. $\sigma(B)$, the embedded base surface), one finds that $A_{\Lambda^2 V}$ is nevertheless still a generic member of its own linear system in B and so nonsingular.

As this point is relevant¹⁹ let us consider this in detail. Assume we deform $A_{\Lambda^2 V}$ to a curve²⁰ $\overline{A}_{\Lambda^2 V}$: can we find a deformed surface²¹ \overline{C}_V with $\overline{A}_{\Lambda^2 V} = \overline{C}_{\Lambda^2 V} \cap \sigma$ where $\overline{C}_{\Lambda^2 V}$ is derived in the usual way from \overline{C}_V (that is by building fibrewise the $p_i + p_j$ for $i < j$)? If true, the generic curve of class $(n-2)\eta - \frac{n(n-1)}{2}c_1$ arises indeed as an $A_{\Lambda^2 V} = C_{\Lambda^2 V} \cap \sigma$ from a generic C_V . The question has two parts: 1.) whether $\overline{A}_{\Lambda^2 V}$ can be understood as $S \cap \sigma$ for a surface S of class $\frac{n(n-1)}{2}\sigma + (n-2)\eta$, and 2.) whether S is a $\overline{C}_{\Lambda^2 V}$.

Concerning 1.) note that among the moduli of possible surfaces S (essentially the parameters in the corresponding equation) there is a subset N of parameters which arises when one considers the ensuing equation for $S \cap \sigma$; there is no obstruction to complete *arbitrary* parameters of N (for the equation of the curve $S \cap \sigma$) to a complete set of parameters for (an equation of) S . For 2.) note that the parameter restriction²² which assures that a general S is a $\overline{C}_{\Lambda^2 V}$ concerns only 'vertical' parameters whereas N refers to 'horizontal' parameters (suggesting that the latter remain unrestricted and generic).

¹⁹in the case of double points of $A_{\Lambda^2 V}$ the resulting correction effect would cancel out, cf. sect. 5.2

²⁰still in the same cohomology class $(n-2)\eta - \frac{n(n-1)}{2}c_1$ in $H^2(B, \mathbf{Z})$

²¹still in the cohomology class $n\sigma + \eta$ in $H^2(X, \mathbf{Z})$ (and with fibre points adding to zero)

²²that the $\frac{n(n-1)}{2}$ fibre points q_k of S arise as $p_i + p_j$ (with $i, j = 1, \dots, n$ and $i < j$) where $\sum p_i = 0$

5.2 The impact of the special points

For a proper *bookkeeping* of the effect of the special points let us define *formally* a corrected Euler number of the curve $A_{\Lambda^2 V}$ which includes besides the ordinary expression from adjunction (for a generic smooth curve of the class of $A_{\Lambda^2 V}$) also a correction term

$$e_{A_{\Lambda^2 V}}^{corr} = e_{A_{\Lambda^2 V}}^{ord} - 2\Delta_n \quad (5.25)$$

With the inclusion of this correction term both expressions in question now come out correctly: on the one hand the Euler number computation for the curve $\tilde{A}_{\Lambda^2 V}$ from the ramified covering $\pi : \tilde{A}_{\Lambda^2 V} \rightarrow A_{\Lambda^2 V}$

$$e_{\tilde{A}_{\Lambda^2 V}} = 2e_{A_{\Lambda^2 V}}^{corr} - \deg Br \quad (5.26)$$

and the generation number computation from the net amount of matter on the other

$$h^1(\Lambda^2 V) - h^1(\Lambda^2 \bar{V}) = \deg(L_{\Lambda^2 V} \otimes K_B)|_{A_{\Lambda^2 V}} + \frac{1}{2}e_{A_{\Lambda^2 V}}^{corr} \quad (5.27)$$

Let us emphasize that the actual motivation of the correction in the Euler number came from the non-standard behaviour of the covering $\pi : \tilde{A}_{\Lambda^2 V} \rightarrow A_{\Lambda^2 V}$ at the special points. Taking into account this non-standard behaviour one has of course to go similarly now through the matter computation of the generation number from the beginning and to find that the final formula *turns out* again to be the previous original formula (for the nonsingular case), now just with the 'renormalized' corrected Euler number (instead of the ordinary one); that actually all the effects of the special points for *this* computation can be absorbed just in a redefined Euler number would deserve a detailed reasoning. What we have shown is that this is so *effectively*.

Having seen that the redefinition (5.25) works let us now explain why it should do so. The crucial identification of the correction term Δ_n is expressed in the relation (cf. (5.20))

$$\boxed{-2\Delta_n = \delta = r_{\Lambda^2 V} \cdot \sigma - \left(\tilde{A}_{\Lambda^2 V} \cdot r_V - \tilde{A}_{\Lambda^2 V} \cdot \sigma_{II} \right)} \quad (5.28)$$

From the interpretation in (5.11) and (5.22) one finds that this non-negative number is the number of special points of $A_{\Lambda^2 V}$ with four rather than only the generic two preimages in the curve $\tilde{A}_{\Lambda^2 V}$ upstairs. The reason for the (positive) occurrence of δ in the *effective correction* (5.25), which then occurs in the correct double covering formula (5.26), is just that over these special points of the curve $A_{\Lambda^2 V}$ downstairs lie not two but four points in the curve $\tilde{A}_{\Lambda^2 V}$ upstairs. Or, more formally: if the mentioned locus of points is taken out on the left hand side of the double covering formula with this fourfold multiplicity and on the right hand side with simple multiplicity (as correction to the Euler number,

i.e. inside the bracket which is multiplied by the covering degree) one arrives *effectively* indeed at the corrected version of the formula as given in (5.26)

$$e_{\tilde{A}_{\Lambda^2 V}} - 4\delta = 2(e_{A_{\Lambda^2 V}}^{ord} - \delta) - \deg Br \quad (5.29)$$

$$\implies e_{\tilde{A}_{\Lambda^2 V}} = 2(e_{A_{\Lambda^2 V}}^{ord} + \delta) - \deg Br \quad (5.30)$$

We emphasize again (cf. footn. 19) that it is decisive that the special points are, despite first appearance perhaps, not double points (but rather ramification points of local branches of the curve $A_{\Lambda^2 V}$ in the base). If they would have been double points one would have had $e_{A_{\Lambda^2 V}}^{ord} = e_{A_{\Lambda^2 V}}^{standard} - \delta$ and the whole effect would have cancelled out (here 'standard' refers to the standard formula from the adjunction computation which only for a smooth curve gives the ordinary Euler number).

This reasoning explains the correction in the double covering formula (5.26) which compares the Euler numbers of the curves $\tilde{A}_{\Lambda^2 V}$ upstairs and $A_{\Lambda^2 V}$ downstairs (the auxiliary problem). We recall that, by contrast, the fact that just this effective correction occurs also again in the computation (5.27) of the generation number from the net amount of matter (the original problem), would deserve a second start and going through that computation while taking into account the influences of the mentioned special points. We stress that we do *not* claim that the correct matter computation *arises* from the use of a corrected Euler number (as causation); rather one has also to review (3.14). Our point is that the complicated inner structure of the correction term, which was completely explained in the auxiliary problem, suggests that this explanation (the uncovered geometric subtlety) captures also the reason for the identical correction in the original problem.

Remark: One might phrase the whole phenomenon by starting from the algebraic-geometric object (more general than a curve) given by the curve $A_{\Lambda^2 V}$ together with some of its points having higher (double) multiplicity; *effectively* then the results (5.26) and (5.27) work out correctly just with the Euler number contribution $e_{A_{\Lambda^2 V}}^{for} + \delta$ of this more subtle object. That this second effect in (5.27) also takes place appropriately (as demanded by the result of the Chern class computation) is now however clearly suggested.

5.3 A second look at the cases $n = 4$ and $n = 5$

Having understood how even²³ in the cases $n \geq 6$ special points arise which can explain the deviations from the expected expressions in the mentioned formulae, let us now come back and have a second look at the special, but physically important cases $n = 4$ and 5 .

²³where (in contrast to the cases $n = 4$ or 5) our previous reasoning just from the relation $\sum_{i=3}^n p_i = 0$ in a fibre over a point of our curve $A_{\Lambda^2 V}$ was no longer sufficient to capture the special points

5.3.1 The case $n = 4$ again

In the case $n = 4$ the general fibre relation $p_1 + p_2 + p_3 + p_4 = 0$ in C_V shows immediately the nonreduced character of the curve $A_{\Lambda^2 V}$ as we described earlier in sect. 4.3. This will, in this special case, modify of course the considerations done for the general case $n \geq 6$. We have two points to tackle: first the discrepancy between the generally computed number δ of special points (on the one hand) and the true nature of the special locus as being given here by the curve $A_{\Lambda^2 V}$ as a whole (on the other hand); and secondly how the appropriate relation between the Euler numbers of $\tilde{A}_{\Lambda^2 V}$ and $A_{\Lambda^2 V}$ is constructed, especially given the two facts of $A_{\Lambda^2 V}$ being non-reduced and the special locus consisting of the base curve $A_{\Lambda^2 V}$ as a whole (instead of being given by a finite set of special points).

Let us start with the first point, the proper adjustment of δ . We had found a proportionality between $\delta_4 = -2\Delta_4 = -2(\eta - 3c_1)c_1$ and the 'special' locus, here actually the curve $A_{\Lambda^2 V}$ (of cohomology class $2(\eta - 3c_1)$) as a whole. Let us now point out where the general argument (for $n \geq 6$) fails that δ gives the cardinality of the locus of special points. One finds in this case that the ramification curve $r_{\Lambda^2 V}$ of the covering $\pi_{\Lambda^2 V} : C_{\Lambda^2 V} \rightarrow B$ is actually reducible (with one component being furthermore non-reduced as $A_{\Lambda^2 V} = 2A_{\Lambda^2 V}^{red}$):

$$\begin{aligned} r_{\Lambda^2 V} &= r_{\Lambda^2 V}^{(\delta)} + r_{\Lambda^2 V}^{(\epsilon)} \\ &= \sigma_{II}|_{C_{\Lambda^2 V}} + r_{\Lambda^2 V}^{(\epsilon)} \\ &= A_{\Lambda^2 V}\sigma + \sigma_2|_{C_{\Lambda^2 V}} + r_{\Lambda^2 V}^{(\epsilon)} \end{aligned} \tag{5.31}$$

We can here distinguish a δ -component $r_{\Lambda^2 V}^{(\delta)}$ and an ϵ -component $r_{\Lambda^2 V}^{(\epsilon)}$ of $r_{\Lambda^2 V}$, being given by the cases of $p_i + p_j = p_k + p_l$ (where $i \neq j$ and $k \neq l$, and of course $\{i, j\} \neq \{k, l\}$) with either $\{i, j\} \cap \{k, l\} = \emptyset$ or $\{i, j\} \cap \{k, l\} \neq \emptyset$, respectively (cf. an analogous remark after (5.20)): in the first case one finds with $p_1 + p_2 = p_3 + p_4$ that $p_1 + p_2 \in \sigma_{II}|_{C_{\Lambda^2 V}}$ (because one also has $p_1 + p_2 = -(p_3 + p_4)$), which itself has the two components $\sigma|_{C_{\Lambda^2 V}} = A_{\Lambda^2 V}\sigma$ and $\sigma_2|_{C_{\Lambda^2 V}}$; in the second case one gets a remaining component of class $r_{\Lambda^2 V}^{(\delta)} - r_{\Lambda^2 V}^{(\epsilon)}$

$$\begin{aligned} r_{\Lambda^2 V}^{(\epsilon)} &= (C_{\Lambda^2 V} + c_1)|_{C_{\Lambda^2 V}} - \sigma_{II}|_{C_{\Lambda^2 V}} \\ &= 2(\sigma + \eta - c_1)|_{C_{\Lambda^2 V}} \\ &= 4[(4\eta - 6c_1)\sigma + \eta^2 - \eta c_1] \end{aligned} \tag{5.32}$$

Here one finds therefore, as we expected above to find in this special case $n = 4$, for the set $\{b \in B \mid p_i + p_j = 0 = p_k + p_l \text{ where } i \neq j, k \neq l \text{ and } \{i, j\} \cap \{k, l\} = \emptyset\}$ (cf. (5.18))

just the reduced component $A_{\Lambda^2 V}^{red}$; as an additional numerical check one convinces oneself that $r_{\Lambda^2 V}^{(\epsilon)} \cdot \sigma = 4(\eta - 3c_1)(\eta - 2c_1)$ such that one has indeed

$$\epsilon = r_{\Lambda^2 V}^{(\epsilon)} \cdot \sigma \quad (5.33)$$

Actually the situation here is still somewhat more complicated, however. The proper counting turns out in this case to employ the halved quantities instead of the general formal expressions: for $r_{\Lambda^2 V}^{(\epsilon)} \cdot \sigma$ one has to use it halved because it comes with double multiplicity (like $2A_{\Lambda^2 V}^{red}\sigma$), and similarly for ϵ the argument given after (5.16) is no longer true such that one has actually to use $\epsilon^{true} := \frac{1}{2}\epsilon$. After these adjustments it is still true, however, that

$$\epsilon^{true} = \frac{1}{2} r_{\Lambda^2 V}^{(\epsilon)} \cdot \sigma \quad (5.34)$$

For ϵ^{true} the argument for the halving is that the cardinality of the subset $\pi(\tilde{A}_{\Lambda^2 V} \cap r_V) \setminus \pi(\tilde{A}_{\Lambda^2 V} \cap \sigma_{II}) = \pi(\tilde{A}_{\Lambda^2 V} \cap r_V \setminus \tilde{A}_{\Lambda^2 V} \cap \sigma_{II})$ of $\pi(\tilde{A}_{\Lambda^2 V} \cap r_V)$ is $\frac{1}{2}(\tilde{A}_{\Lambda^2 V} \cdot r_V - \tilde{A}_{\Lambda^2 V} \cdot \sigma_{II})$ because the case that p_3 is p_1 , say (the case of $p_2 = p_1$ was excluded here), entails that also $p_4 = -p_3 = -p_1 = p_2$; thus one has two intersections above in $\tilde{A}_{\Lambda^2 V} \cap r_V \setminus \tilde{A}_{\Lambda^2 V} \cap \sigma_{II}$ over one and the same point $b \in A_{\Lambda^2 V} \subset B$ below. For the same reason the other object, $r_{\Lambda^2 V}^{(\epsilon)} \cdot \sigma$ comes with an inherent doubled multiplicity (relative to the set-theoretic version).

This resolves the first discrepancy in this case of $n = 4$: the special locus is the base curve as a whole and the appropriate refined consideration of $r_{\Lambda^2 V}$ is in harmony with the proper δ - and ϵ -computations (the formal expression with $\delta_4 < 0$ causes no problem).

Now we have to tackle the second point: the Euler number computation for the covering $\pi : \tilde{A}_{\Lambda^2 V} \rightarrow A_{\Lambda^2 V}$. The standard interpretation of the relation (5.30), which *as a purely numerical relation* is still true also for $n = 4$, is no longer appropriate now, of course. The proper interpretation follows now the following computation

$$e_{\tilde{A}_{\Lambda^2 V}} = 4e_{A_{\Lambda^2 V}^{red}} - 2\epsilon^{true} - \tilde{A}_{\Lambda^2 V} \cdot \sigma_{II} \quad (5.35)$$

where

$$e_{\tilde{A}_{\Lambda^2 V}} = -4(\eta - 3c_1)(3\eta - 7c_1) \quad (5.36)$$

$$e_{A_{\Lambda^2 V}^{red}} = -(\eta - 3c_1)(\eta - 4c_1) \quad (5.37)$$

$$\epsilon^{true} = 2(\eta - 3c_1)(\eta - 2c_1) \quad (5.38)$$

$$\tilde{A}_{\Lambda^2 V} \cdot \sigma_{II} = 4(\eta - 3c_1)(\eta - c_1) \quad (5.39)$$

Here one has a more complicated ramification structure: in addition to the usual simple ramification related to $\tilde{A}_{\Lambda^2 V} \cap \sigma_{II}$ (where two of the now four branches of $\tilde{A}_{\Lambda^2 V}$ come

together) one has now also the double ramification related to a number of ϵ^{true} points in the base curve $A_{\Lambda^2 V}^{red}$ over which two times two of the four branches come together (these points also turn out to be ramification points, not double points): this represents the coincidences from $p_3 = p_1$, say, (and so also $p_4 = p_2$) which are typical for the ϵ contribution.

5.3.2 The case $n = 5$ again

Things are different in many respects in the case $n = 5$. In that case we had pointed out in sect. 4.3 that the special locus we had found (as the possible source of the deviation) was not yet precisely the source of the correction Δ_5 as the latter was not proportional to ν_5 . This discrepancy is explained, however, easily as follows. To have in a fibre of C_V four points p_i, p_j, p_k, p_l with $p_i + p_j = 0$ (where $i \neq j$) and $p_k + p_l = 0$ (where $k \neq l$) *with all four indices i, j, k, l different*²⁴ one has to make sure that the point playing the role of p_5 (in the argument given above in sect. 4.3) was not already among the pair consisting of the points p_1 and p_2 (which led us to a generic point of the curve $A_{\Lambda^2 V}$ in the first instance; so here $\{1, 2\} = \{i, j\}$); the latter possibility (of $p_5 = p_1$, say, where then in fact $p_1 = 0 = p_2$ because we imposed the condition $p_5 = 0$ in the argument of sect. 4.3) corresponds to the locus $\tilde{A}_{\Lambda^2 V} \cap \sigma$ (which therefore has to be taken out) of cardinality $\tilde{A}_{\Lambda^2 V} \cdot \sigma = \eta^2 - 9\eta c_1 + 20c_1^2$. One finds $\nu_5 - \tilde{A}_{\Lambda^2 V} \cdot \sigma$ as cardinality of the 'proper' special locus and realizes furthermore that (for δ_5 cf. (5.18) and (5.20))

$$\delta_5 = \nu_5 - \tilde{A}_{\Lambda^2 V} \cdot \sigma \quad (5.40)$$

restoring thus the proportionality to Δ_5 (as one has, as always, $\delta_5 = -2\Delta_5$).

So here, in contrast to the case $n = 4$, the curve $A_{\Lambda^2 V}$ is reduced and a proper computation, even along the lines of sect. 4.3 (avoiding the more general considerations from sect. 5.1), gives the correct number of special points (this corresponds to the first point we had to tackle in the case $n = 4$); so the interpretation of the Euler number computation for the covering $\pi : \tilde{A}_{\Lambda^2 V} \rightarrow A_{\Lambda^2 V}$ given in sect. 5.2 (whose adjustment constituted the second point we had to treat for $n = 4$) remains completely intact here.

Needless to say, the cautionary remarks made at the end of sect. 5.2 concerning an extension of the analysis of the impact of the locus of special points from the analogous auxiliary problem (of the Euler number computation) to the proper original problem (of the matter computation) remain in order also for the cases $n = 4$ and 5.

²⁴the meaning of the cases where $\{i, j\} \neq \{k, l\}$ but $\{i, j\} \cap \{k, l\} \neq \emptyset$ was investigated in sect. 5.1

6 Conclusion

A particularly relevant case of heterotic model building concerns supersymmetric GUT models with GUT group $G = SU(5)$ or $SO(10)$ (for a list containing some of the many important contributions in this field cf. [4]). These cases correspond to the structure group H_V of the heterotic bundle V being $SU(5)$ and $SU(4)$, resp.. Consideration of the number of matter multiplets leads to computation of the bundle cohomology of V and $\Lambda^2 V$ (or complex conjugates); the case of the Higgses leads to consideration of the latter.

In the framework of the spectral cover approach [1] to $SU(n)$ bundles on an elliptic Calabi-Yau threefold $\pi : X \rightarrow B$ one considers an n -fold ramified cover C_V of B (together with a line bundle on C_V which, in the usual construction, is essentially fixed). The intersection curve $A_V := C_V \cap B$ plays an important role: cohomological computations of V over X are reduced to corresponding computations for a line bundle over A_V .

For the computation of the number of Higgses thus the curve $A_{\Lambda^2 V}$ is particularly important²⁵: as mentioned one needs to compute the Chern class of the relevant line bundle $L_{\Lambda^2 V}$, at least when restricted to this intersection curve; this is formula²⁶ (3.14), which uses the double cover relation of $\tilde{A}_{\Lambda^2 V}$ and $A_{\Lambda^2 V}$. We will consider just the *generic case* of this set-up (whereas [3] treats in detail a *specific example*).

Evaluating now this computation on the matter curve $A_{\Lambda^2 V}$ just for the net amount $h^1(\Lambda^2 V) - h^1(\Lambda^2 \bar{V})$ of the corresponding matter one finds, cf. (4.5), an additional ('correction') term Δ_n relative to the expectation from the index computation from $c_3(\Lambda^2 V)$.

As this surprising deviation is puzzling we switch to an auxiliary problem²⁷ which involves the same double covering $\pi : \tilde{A}_{\Lambda^2 V} \rightarrow A_{\Lambda^2 V}$: the comparison of Euler numbers of the two curves (taking into account the ramification). We find again a deviation from the ordinarily expected relation, and again with just the expression Δ_n as correction.

We can explain, by a geometric subtlety which constitutes a deviation (details are in sect. 5) from the expected standard covering picture for $\pi : \tilde{A}_{\Lambda^2 V} \rightarrow A_{\Lambda^2 V}$, the mismatch in the latter problem completely, i.e. we can derive just the needed correction. As the correction term itself has a relatively complicated parametric structure we argue that this captures already the rationale behind the (identical) correction in the original problem.

²⁵From the perspective of the original bundle V itself this is the locus of points in B where (essentially) in the corresponding fibre of C_V two points ($i \neq j$) fulfil $p_i + p_j = 0$. Thus $A_{\Lambda^2 V}$ has as double cover the non-trivial divisor component $\tilde{A}_{\Lambda^2 V}$ in $C_V \cap \tau C_V$ (the other components being given by $C_V \cap \sigma_{II}$).

²⁶i.e. (138) of [3], relying on (133) which assumes $\pi : \tilde{A}_{\Lambda^2 V} \rightarrow A_{\Lambda^2 V}$ to be an ordinary (ramified) covering

²⁷where (3.14) is not concerned but only the assumptions on which it itself relies, that $\pi : \tilde{A}_{\Lambda^2 V} \rightarrow A_{\Lambda^2 V}$ is an ordinary (ramified) double covering between generically (irreducible, reduced and) smooth curves

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